Stretched exponential relaxation in the mode-coupling theory for the Kardar-Parisi-Zhang equation

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We study the mode-coupling theory for the Kardar-Parisi-Zhang equation in the strong-coupling regime, focusing on the long time properties. By a saddle point analysis of the mode-coupling equations, we derive exact results for the correlation function in the long-time limit—a limit that is hard to study using simulations. The correlation function at wave vector $\mathbf{k}$ in dimension d is found to behave asymptotically at time t as $C(\mathbf{k},t) \simeq A/k^{d+4-2z}(Btk^z)^{\gamma/z} \exp[-(Btk^z)^{1/z}]$, with $\gamma=(d-1)/2$, A a determined constant, and B a scale factor.

The Kardar-Parisi-Zhang (KPZ) [1] equation is a simple nonlinear Langevin equation, proposed in 1986 as a coarse-grained description of a growing interface. Probably due to the fact that is the simplest generalization of the diffusion equation, which includes a relevant nonlinear term, the KPZ equation also arises in connection with many other important physical problems (the Burgers equation for one-dimensional turbulence [2], directed polymers in a random medium [3–5] etc.).

The KPZ equation for a growing interface, described by a single valued height function $h(\mathbf{x},t)$ on a d-dimensional substrate $\mathbf{x} \in \mathfrak{R}^d$ is

$$\partial_t h(\mathbf{x},t) = \nu \nabla^2 h + \frac{\lambda}{2}(\nabla h)^2 + \eta(\mathbf{x},t). \tag{1}$$

The first term represents the surface-tension forces that tend to smooth the interface, the second describes the nonlinear growth locally normal to the surface, and the last is a noise that mimics the stochastic nature of the growth process [6]. We choose the noise to be Gaussian, with zero mean and second moment $\langle \eta(\mathbf{x},t) \eta(\mathbf{x}',t') \rangle = 2D \delta^d(\mathbf{x}-\mathbf{x}') \delta(t-t')$. The steady-state interface profile is usually described in terms of the roughness: $w = \sqrt{\langle h^2(\mathbf{x},t) \rangle - \langle h(\mathbf{x},t) \rangle^2}$, which for a system of size L behaves like $L^\chi f(t/L^z)$, where $f(x) \to$ const as $x \to \infty$ and $f(x) \sim x^{\chi/z}$ as $x \to 0$, so that w grows with time like $t^{\chi/z}$ until it saturates to L^χ when $t \sim L^z$. χ and z are the roughness and dynamic exponent, respectively.

The phenomenology of the KPZ is well known: above two dimensions, there are two distinct regimes, separated by a critical value λ_c of the nonlinearity coefficient. In the weak-coupling regime ($\lambda < \lambda_c$) the nonlinear term is irrelevant and the behavior is governed by the Gaussian ($\lambda = 0$) fixed point. The KPZ in this regime is equivalent to the linear Edward Wilkinson equation, for which the exponents are known exactly $\chi=(2-d)/2$ and $z=2$. The more challenging strong-coupling regime ($\lambda > \lambda_c$), where the nonlinearity is relevant, is characterized by anomalous exponents, which are not known exactly in general dimension d. From the Galilean invariance [2] [invariance of Eq. (1) under an infinitesimal tilting of the surface] one can derive the relation $\chi + z = 2$, which leaves just one independent exponent. For the special case when $d=1$, the existence of a fluctuation-dissipation theorem gives the exact results $\chi=1/2$, $z=3/2$.

We take a self-consistent approach, the mode-coupling (MC) approximation [7,8], in which in the diagrammatic expansion for the correlation and response function, only diagrams that do not renormalize the three point vertex λ are retained. The MC approximation has been remarkably successful in other areas of condensed-matter physics, for example, in the study of structural glass transitions [9], binary mixtures [10], and critical dynamics of magnets [10,11]. KPZ mode-coupling reproduces the exact values of the exponents in $d=1$. Furthermore, MC equations have been shown to arise from the large N limit of a generalized N-component KPZ equation [8], which allows, in principle, a systematic approach to the theory beyond mode coupling, by expanding in $1/N$.

Our study focuses on the large-time properties of the KPZ equation; we predict a stretched exponential decay for the correlation function at large times. A phenomenological stretched exponential law was used as long ago as 1854 to fit electronic relaxation data for a Leiden jar capacitor [12], and has been rediscovered many times; fitting functions involving stretched exponentials are nowadays widely used in the phenomenogical analysis of relaxation data (examples are dielectric relaxation [13] and glassy relaxation [9,14]). However, only a few analytical arguments [15] are able to reproduce stretched exponential relaxation in complex systems. Our prediction is in principle amenable of numerical verification, even though usual numerical techniques, mainly based on simulations [16] of discrete microscopical models that belong to the KPZ universality class [17], are hard in the long-time asymptotic region.

Mode-coupling equations are coupled equations for the correlations and response function. The correlation and response functions are defined in Fourier space by

$$C(\mathbf{k},\omega) = \langle h(\mathbf{k},\omega) h^*(\mathbf{k},\omega) \rangle,$$

$$G(\mathbf{k},\omega) = \delta^{-d}(\mathbf{k}-\mathbf{k}') \delta^{-1}(\omega-\omega') \langle \partial h(\mathbf{k},\omega)/\partial \eta(\mathbf{k}',\omega') \rangle,$$

where $\langle \cdot \rangle$ indicates an average over η. In the mode-coupling approximation, the correlation and response functions are the solutions of two coupled equations,

$$G^{-1}(\mathbf{k},\omega)=G_0^{-1}(\mathbf{k},\omega)+\lambda^2\int\frac{d\Omega}{2\pi}\int\frac{d^dq}{(2\pi)^d}$$

$$\times[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})][\mathbf{q}\cdot\mathbf{k}]G(\mathbf{k}-\mathbf{q},\omega-\Omega)C(\mathbf{q},\Omega),$$

$$(2)$$

$$C(\mathbf{k},\omega)=C_0(\mathbf{k},\omega)+\frac{\lambda^2}{2}|G(\mathbf{k},\omega)|^2\int\frac{d\Omega}{2\pi}\int\frac{d^dq}{(2\pi)^d}$$

$$\times[\mathbf{q}\cdot(\mathbf{k}-\mathbf{q})]^2C(\mathbf{k}-\mathbf{q},\omega-\Omega)C(\mathbf{q},\Omega),\quad(3)$$

where $G_0(\mathbf{k},\omega)=(\nu k^2-i\omega)^{-1}$ is the bare response function, and $C_0(\mathbf{k},\omega)=2D|G(\mathbf{k},\omega)|^2$. In the strong-coupling limit, $G(\mathbf{k},\omega)$ and $C(\mathbf{k},\omega)$ take the following scaling forms

$$G(\mathbf{k},\omega)=k^{-z}g(\omega/k^z),$$

$$C(\mathbf{k},\omega)=k^{-(2\chi+d+z)}n(\omega/k^z),$$

and Eqs. (2) and (3) translate into the following coupled equations for the scaling functions $n(x)$ and $g(x)$:

$$g^{-1}(x)=-ix+I_1(x),\qquad(4)$$

$$n(x)=|g(x)|^2I_2(x),\qquad(5)$$

where $x=\omega/k^z$ and $I_1(x)$ and $I_2(x)$ are given by

$$I_1(x)=P\int_0^\pi d\theta\sin^{d-2}\theta\int_0^\infty dq\cos\theta(\cos\theta-q)q^{2z-3}r^{-z}$$

$$\times\int_{-\infty}^\infty dy\,g\left(\frac{x-q^zy}{r^z}\right)n(y),$$

$$I_2(x)=\frac{P}{2}\int_0^\pi d\theta\sin^{d-2}\theta\int_0^\infty dq(\cos\theta-q)^2q^{2z-3}r^{-(d+4-z)}$$

$$\times\int_{-\infty}^\infty dy\,n\left(\frac{x-q^zy}{r^z}\right)n(y),$$

with $P=\lambda^2/(2^d\Gamma[(d-1)/2]\pi^{(d+3)/2})$, $r^2=1+q^2-2q\cos\theta$. It will be convenient to write Eqs. (4) and (5) as a function of time t, by Fourier transforming them

$$\frac{\widehat{g_R}}{|g|^2}(t)=\hat{I}_1(t),\qquad(6)$$

$$\frac{\hat{n}}{|g|^2}(t)=\hat{I}_2(t),\qquad(7)$$

where $\hat{I}_1$ is the Fourier transform of the real part of I_1 and $\hat{I}_2$ is the Fourier transform of I_2

$$\hat{I}_1(t)=2\pi P\int_0^\pi d\theta\sin^{d-2}\theta$$

$$\times\int_0^\infty dq\cos\theta(\cos\theta-q)q^{2z-3}\widehat{g_R}(tr^z)\hat{n}(tq^z),$$

$$\hat{I}_2(t)=\pi P\int_0^\pi d\theta\sin^{d-2}\theta$$

$$\times\int_0^\infty dq(\cos\theta-q)^2q^{2z-3}r^{-(d+4-2z)}\hat{n}(tr^z)\hat{n}(tq^z).$$

We will show that an asymptotic solution $\hat{n}(t)=\widehat{n_\infty}(t)$ for $t\to\infty$ of Eq. (7) is given by

$$\widehat{n_\infty}(t)=A(Bt)^{\gamma/z}e^{-|Bt|^{\alpha/z}},\qquad(8)$$

with $\gamma=(d-1)/2$, $\alpha=1$, and

$$A=\frac{g(0)^{-2}4\Gamma(4z-4)}{P2^{(d-1)/2}\Gamma[(d-1)/2]\Gamma(2z-2)^2}.\qquad(9)$$

Before we proceed with the details of the calculation, we shall briefly outline the strategy: assuming a solution of the form (8) we evaluate the right-hand side of Eq. (7) in the large t limit by saddle-point methods. The result of this analysis shows that any $\alpha\le1$ allows us to reproduce the exponential factor on the left-hand side. For $\alpha\le z$, we find that the left-hand side of Eq. (7) is dominated by $g(0)^{-2}\hat{n}(t)$ (see below), thus the asymptotic form of $I_2(t)$ has to be matched with $g(0)^{-2}\widehat{n_\infty}(t)$. While the exponential factor can be matched by any $\alpha\le1$, the only value of α that also allows us to match the power-law factor $t^{\gamma/z}$ is $\alpha=1$. We can then proceed with $\alpha=1$ to get $\gamma=(d-1)/2z$ and A given by Eq. (9). Note that even though the scaling part of the correlation and response function are coupled in the two Eqs. (6) and (7), the knowledge of the specific form of $\hat{g}(t)$ is not required to extract the large t behavior of $\hat{n}(t)$.

Now we proceed with the details of the saddle-point calculation. Let us start with the evaluation of the left-hand side of Eq. (7). Expanding $|g(x)|^2$ in even powers of x around $x=0$, we can write $\widehat{(n/|g|^2)}(t)=a_0\hat{n}(t)+a_1d^2\hat{n}(t)/dt^2$ $+\ldots$, where $a_0=g(0)^{-2}$. For $\hat{n}(t)$ of the form (8) with $\alpha\le z$, the first term of the series will dominate at large t, we can therefore assume that the left-hand side of Eq. (7) has the asymptotic behavior $\widehat{(n/|g|^2)}(t)\simeq g(0)^{-2}\hat{n}(t)$. Let us now turn to the asymptotic analysis of $\hat{I}_2(t)$. First note that the integral $\hat{I}_2(t)$, is symmetric in the exchange $q\to r$, $\theta\to\phi$, where $r\sin\phi=q\sin\theta$ (see Fig. 1). This allows us to rewrite $\hat{I}_2(t)$ as twice the integral restricted to the region $q\cos\theta<1/2$:
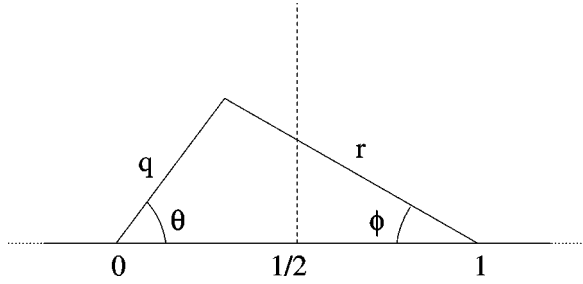
FIG. 1. Region of integration for $\hat{I}_2(t)$ in Eq. (10).

$$\hat{I}_2(t)=2\pi P\int_0^\infty dq\int_{\bar{\theta}}^\pi d\theta\sin^{d-2}\theta$$

$$\times\int_0^\infty dq(\cos\theta-q)^2 q^{2z-3}r^{-(d+4-2z)}\hat{n}(tr^z)\hat{n}(tq^z),\tag{10}$$

where $\cos\bar{\theta}=\max[1/2q,1]$. In $\hat{I}_2(t)$ the function $\hat{n}(t)$ appears in the integrand with the arguments tq^z, tr^z. For large enough t, we can always safely replace $\hat{n}(tr^z)$ with its asymptotic form $\widehat{n_\infty}(tr^z)$, since in the new region of integration $r\geq 1/2$. We have to use more care with $\hat{n}(tq^z)$, since such replacement is not allowed for all q, but only for $q>C/t^{1/z}$, where C is a large constant. We then have $\hat{I}_2(t)\simeq\widehat{I_2^\infty}(t)=J_1(t)+J_2(t)$, where

$$J_1(t)=2\pi P\int_0^{C/t^{1/z}}dq\int_{\bar{\theta}}^\pi d\theta\sin^{d-2}\theta$$

$$\times(\cos\theta-q)^2 q^{2z-3}r^{-(d+4-2z)}\widehat{n_\infty}(tr^z)\hat{n}(tq^z),$$

$$J_2(t)=2\pi P\int_{C/t^{1/z}}^\infty dq\int_{\bar{\theta}}^\pi d\theta\sin^{d-2}\theta$$

$$\times(\cos\theta-q)^2 q^{2z-3}r^{-(d+4-2z)}\widehat{n_\infty}(tr^z)\widehat{n_\infty}(tq^z).$$

The contribution from J_1 is negligible with respect to J_2 as $t\to\infty$. Let us evaluate an asymptotic expression for J_2 and postpone the discussion of J_1 to the end. Inserting $\widehat{n_\infty}(t)=A(Bt)^{\gamma/z}\exp(-|Bt|^{\alpha/z})$ in $J_2(t)$ gives

$$J_2(t)=2\pi PA^2(Bt)^{2\gamma/z}\int_{C/t^{1/z}}^\infty dq\int_{\bar{\theta}}^\pi d\theta\sin^{d-2}\theta$$

$$\times(\cos\theta-q)^2 q^{2z-3+\gamma}r^{-(d+4-2z)+\gamma}$$

$$\times\exp[-(Bt)^{\alpha/z}(q^\alpha+r^\alpha)].\tag{11}$$

It is immediately clear that the main contribution to this integral will come from the region where $f(q,\theta)=q^\alpha+r(q,\theta)^\alpha$ has its minimum, i.e., from the segment $\theta=0$, $C/t^{1/z}<q<1/2$ (for large enough t, $C/t^{1/z}<1/2$). The value of q which minimizes f will depend on α: for $\alpha<1$, f reaches its minimum at $q_0=C/t^{1/z}$, where $f(q_0,0)=C^\alpha/t^{\alpha/z}+|(1-C/t^{1/z})|^\alpha\simeq 1+(C/t^{1/z})^\alpha+O(C/t^{1/z})$. For α

>1, the minimum is realized at $q=1/2$, where $f(1/2,0)=1/2^{\alpha-1}$. For $\alpha=1$, all q in the region $C/t^{1/z}<q<1/2$ contribute equally, giving $f(q,0)=1$. The saddle-point approximation in the angular integral gives

$$J_2(t)\simeq 2\pi PA^2(Bt)^{2\gamma/z}\int_{C/t^{1/z}}^{1/2}dq$$

$$\times\exp[-(Bt)^{\alpha/z}(q^\alpha+|1-q|^\alpha)]$$

$$\times|1-q|^{2-(d+4-2z)+\gamma}q^{2z-3+\gamma}\int_0^\pi d\theta\theta^{d-2}$$

$$\times\exp\left[-(Bt)^{\alpha/z}\frac{\alpha}{2}q|1-q|^{\alpha-1}\theta^2\right].\tag{12}$$

Performing the integral over θ in the large t limit

$$J_2(t)\simeq\pi PA^2\Gamma\left(\frac{d-1}{2}\right)\left(\frac{2}{\alpha}\right)^{d-1/2}(Bt)^{2\gamma/z-(d-1)/2z}$$

$$\times\int_{C/t^{1/z}}^{1/2}dq\exp[-(Bt)^{\alpha/z}(q^\alpha+|1-q|^\alpha)]|1-q|^\phi q^\psi,\tag{13}$$

where $\phi=2z-d-2+\gamma+(1-\alpha)(d-1)/2$ and $\psi=2z-3+\gamma-(d-1)/2$. The leading contribution for large t is then given by

$$J_2(t)\simeq\pi PA^2\Gamma\left(\frac{d-1}{2}\right)\left(\frac{2}{\alpha}\right)^{d-1/2}(Bt)^{2\gamma/z-(d-1)/2z}$$

$$\times\begin{cases}\dfrac{\Gamma((\psi+1)/\alpha)}{\alpha}e^{-(Bt)^{\alpha/z}}(Bt)^{-(\psi+1)/z} & \alpha<1\\[2mm] I\,e^{-(Bt)^{1/z}} & \alpha=1\\[2mm] \sqrt{\pi}2^{-(\phi+\psi)}\exp\left[-\dfrac{(Bt)^{\alpha/z}}{2^{\alpha-1}}\right](Bt)^{-\alpha/2z} & \alpha>1,\end{cases}\tag{14}$$

where $I=\int_0^{1/2}dq\,q^\psi(1-q)^\phi$. For $\alpha\leq 1$, the left-hand side of Eq. (7) will be dominated by a term of the form $g(0)^{-2}\widehat{n_\infty}(t)$, thus $\widehat{I_2(t)}\simeq J_2(t)$ must be proportional to $\widehat{n_\infty}(t)$. This can only be achieved with $\alpha=1$ and $\gamma=(d-1)/2$. For $\alpha<1$, the stretched exponential factor is recovered, but trying to match the power of t leads to $\gamma=(d-1)/2+\psi+1$, which can only be satisfied by the unphysical value $z=1$. For $1<\alpha<z$, the left-hand side of Eq. (7) cannot be matched by J_2. We then conclude that α must be equal to 1, and the asymptotic solution is then given by Eq. (8). The coefficient A can be determined as well by observing that $\gamma=(d-1)/2$ gives $\psi=\phi=2z-3$, and thus $I=\Gamma(2z-2)^2/2\Gamma(4z-4)$, which leads to Eq. (9).

Let us now go back to the integral J_1. The argument of $\hat{n}(tq^z)$ in J_1 runs between 0 and C^z. Thus we can obtain an upper bound on J_1 by replacing $\hat{n}(tq^z)$ by its maximum in this region (which is given by $\hat{n}(0)$ if $\hat{n}(t)$ is a monotoni-

cally decreasing function). An analysis similarly to the one done for J_2, can then be performed for J_1, leading to $J_1 \propto (Bt)^{\gamma/z - (d-1)/2z} e^{-(Bt)^{\alpha/z}}$. Thus the contribution to $\tilde{I_2}$ from J_1 is down by a factor $(Bt)^{-\gamma/z}$.

The scale parameter B is just that, and remains unfixed in terms of the parameters of the KPZ equation. In a recent work [18], we proposed a scaling ansatz for the correlation function as $z \to 2$. If such an ansatz is correct, in a parametrization where $g(0)$ is finite, the scale B^{-1} on which t varies would go to zero as z approaches 2: $B^{-1} \simeq (2 - z)$. Thus the asymptotic solution that we have found here could be read as

an asymptotic solution for all t as z approaches 2.

In summary, we have presented an analytical derivation of an asymptotic solution of the mode-coupling equations for the strong-coupling regime of the KPZ equation in the large t limit. We predict a stretched exponential relaxation for the correlation function, with a power-law prefactor. We hope that this prediction will stimulate numerical investigations of long-time limits generally.

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